TOWARDS A SPECTRAL CHARACTERIZATION OF SIGNALS SUPPORTED ON SMALL-WORLD NETWORKS

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ABSTRACT

We study properties of the family of small-world random graphs introduced in Watts & Strogatz (1998), focusing on the spectrum of the normalized graph Laplacian. This spectrum influences the extent to which a signal supported on the vertices of the graph can be simultaneously localized on the graph and in the spectral domain (the surrogate of the frequency domain for signals supported on a graph). This characterization has implications for inferring or interpolating functions supported on such graphs when observations are only available at a subset of nodes.

1. INTRODUCTION

There is a growing body of work concerned with inferring, filtering, compressing, and otherwise studying signals that are supported on graphs [1]. There are a variety of applications where the data naturally is supported on the vertices of a graph, including social networks, transportation networks, and telecommunication networks. In other applications involving high-dimensional data, the data may naturally lie on a low-dimensional manifold in which case it is common to impose a graph structure over the data as a means of estimating or approximating the structure of the manifold.

The uncertainty principle is a fundamental tenet of signal processing [2]. The classical uncertainty principle states that a continuous-time signal cannot be simultaneously localized in both the time domain and the frequency domain. Agaskar and Lu [3, 4] have recently initiated the study of uncertainty principles for signal supported on graphs. A signal $\boldsymbol{x}$ on a graph $G = (V, E)$ is a vector with one component $x(v) \in \mathbb{R}$ at each vertex $v \in V$ of the graph; typically when studying signal on graphs one expects that the values $x(u)$ and $x(v)$ at two vertices u and v should somehow related (e.g., have similar values) if there is an edge $(u, v) \in E$ connecting them in G (i.e., if u and v are neighbours). Agaskar and Lu [4] define notions of the graph-domain spread $\Delta_g^2(\boldsymbol{x})$ and the spectral-domain spread $\Delta_s^2(\boldsymbol{x})$ of a signal $\boldsymbol{x}$ which both are related to the connectivity of the graph G. The graph spread quantifies the extent to which a signal is spread over the graph as a function of the distance from each node to an appropriately chosen (signal-specific) center vertex. The spectral spread quantifies the extent to which a signal is spread in the spectral domain, where the spectral domain is defined in terms of the eigendecomposition of the graph Laplacian.[1] Based on these notions, one can begin to study the extent to which a signal can be simultaneously localized in both the graph and spectral domains, as a function of the graph topology.

[1] Precise definitions are given in Section 3.

In the present paper we study spectral properties of small-world networks. Small-world networks [5] are networks which simultaneously are highly clustered (i.e., the graph contains many triangles) and have short average path length (the number of hops on the shortest path connecting a randomly drawn pair of vertices). Networks with the small-world property have been observed in a variety of applications, including social networks, biological networks, and telecommunication networks. Watts and Strogatz [5] propose a model for generating small-world networks. Their model begins with a K-connected ring (which is highly clustered, but has long path lengths) and randomly rewires each edge with probability p. The random rewiring reduces the clustering but also dramatically reduces the average path length. Watts and Strogatz [5] observe that for relatively small values of p, the resulting random graphs remain highly clustered while simultaneously having small average path length.

Previously, we reported on a set of experiments looking at the uncertainty curve of small-world networks [6]. The uncertainty curve $\gamma(s)$ is the smallest graph spread $\Delta_g^2(\boldsymbol{x})$ achievable by a signal $\boldsymbol{x}$ with spectral spread $\Delta_s^2(\boldsymbol{x}) = s$. Figure 1 depicts the expected uncertainty curve $\gamma(s)$ for different instances of small world networks with rewiring probability ranging from $p = 0$ to $p = 1$. When $p \approx 0$, the uncertainty curve is very similar to that of a K-connected ring, and when $p \approx 1$, the uncertainty curve is very similar to that of an Erdős-Rényi random graph, where all edges are independent. Interestingly, we observe a sharp transition where, for intermediate values of p, the uncertainty curve undergoes a rapid shift between these two regimes. This shift occurs around values of p where the clustering drops rapidly, and the location of the shift is also well-correlated with the second smallest eigenvalue of the graph Laplacian rapidly increasing [6]. These observations motivate us to further investigate the dependency of the uncertainty curve on the Laplacian spectrum and to study the Laplacian spectrum of small-world networks.

The contributions of this article are as follows. First we show that when the spectral spread is equal to zero, the graph spread cannot be made smaller than $1/2$. Combined with previous results (that the spectral spread is exactly equal to 1 when the spectral spread is 0), this implies that the lower uncertainty curve always remains bounded away from zero at its extremities. Then, towards characterizing the lower uncertainty curve of small-world networks, we study spectral properties of the Laplacian matrix of this family of networks.

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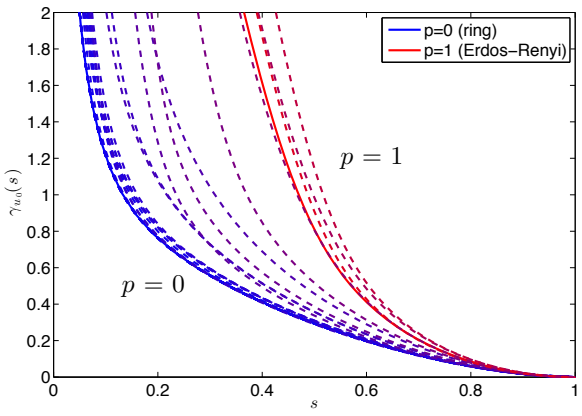

Fig. 1. Lower uncertainty curve of small-world networks for varying values of p from 0 to 1. We observe that the uncertainty curve undergoes a sharp transition between being close to that of a K-connected ring ($p \approx 0$) to being close to that of an Erdős-Rényi random graph [6]. The curves are generated via simulation for networks with $n = 1024$ vertices, with p ranging from 0.001 to 1 on a logarithmic scale.

2. SIGNALS ON GRAPHS

This section begins by introducing concepts and notation that will be used throughout the paper. Let $G = (V, E)$ be a graph with vertex set V and edge set $E \subseteq V \times V$. In this paper we restrict our attention to graphs which are simple, undirected, and unweighted.[2] For a graph with $n = |V|$ vertices, we index the vertices using the integers modulo n; i.e., $V = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$. Consequently, the adjacency matrix $\boldsymbol{A}$ of the graph is symmetric and binary-valued, with $A_{u,v} = A_{v,u} = 1$ if and only if $(u, v) \in E$. The adjacency matrix exactly captures the structure of the graph, in the sense that there is a bijection between G and $\boldsymbol{A}$. The degree of a vertex $v \in V$, denoted by $\deg(v)$, is the number of neighbours of v in G, where the neighbours of a vertex v are those vertices u for which $(u, v) \in E$. Let $\mathbf{1}$ denote the $n \times 1$ vector with all entries equal to 1, and for a vector $\boldsymbol{x} \in \mathbb{R}^n$, let $x(v) = [\boldsymbol{x}]_v$ denote the vth entry of $\boldsymbol{x}$. Then the degree of v is equal to $\deg(v) = [\boldsymbol{A}\mathbf{1}]_v$.

The normalized graph Laplacian is another fundamental matrix which captures properties of the graph G. Understanding properties of a graph through the study of the Laplacian matrix is the central aim of spectral graph theory, and we next recount basic facts from spectral graph theory following [7]. Let $\boldsymbol{D} = \mathrm{diag}(\boldsymbol{A}\mathbf{1})$ denote a diagonal matrix whose vth entry is equal to the degree of vertex v. Then the normalized graph Laplacian is the $n \times n$ matrix

$$\boldsymbol{L} \stackrel{\text{def}}{=} \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2} ,$$

where $\boldsymbol{I}$ denotes the identity matrix. The normalized Laplacian, and its spectrum in particular, captures many properties related to the structure of the graph [7]. Since $\boldsymbol{A}$ is symmetric and $\boldsymbol{D}$ is diagonal, $\boldsymbol{L}$ is also symmetric and so it has an eigenvalue decomposition,

$$\boldsymbol{L} = \boldsymbol{F}\boldsymbol{\Lambda}\boldsymbol{F}^T ,$$

where $\boldsymbol{F}$ is a $n \times n$ orthonormal matrix whose columns are the eigenvectors of $\boldsymbol{L}$, and $\boldsymbol{\Lambda}$ is a diagonal matrix of real eigenvalues.

It is well known that if G is connected then $\boldsymbol{L}$ has one eigenvalue equal to zero, and all other eigenvalues are strictly positive.[3] Moreover, the largest eigenvalue of $\boldsymbol{L}$ is at most 2, and it is equal to 2 if and only if G is bipartite. Throughout the following we assume that G is connected and we denote the eigenvalues of $\boldsymbol{L}$ by

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq 2 .$$

Let $\boldsymbol{F} = [\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-1}]$; i.e., $\boldsymbol{f}_j$ is the jth column of $\boldsymbol{F}$. The eigenvector $\boldsymbol{f}_0$ corresponding to λ_0 can be taken such that all entries are positive, in which case its values are given by

$$f_0(v) = \sqrt{\frac{\deg(v)}{\sum_{u \in V} \deg(u)}} .$$

Since $\boldsymbol{F}$ is an orthogonal matrix, it follows that all other eigenvectors $\boldsymbol{f}_j$, $j \neq 0$, contain both positive and negative entries. In particular, it can be shown that the sign of the entries of the second eigenvector, $\boldsymbol{f}_1$, defines a cut of the graph with interesting properties for partitioning the graph into two sets, V_1 and V_2, which are each well-connected (i.e., a vertex $v \in V_1$ has many neighbours in V_1) and for which the number of edges with one end in V_1 and the other end in V_2 is small [8].

The aim of our study is to understand how the structure of graphs influences the class of signals that are "naturally" supported on the graph. A signal on a graph is a mapping from the vertices V to the reals $\mathbb{R}$. As such, a signal $\boldsymbol{x}$ on a graph can be considered a vector in $\mathbb{R}^n$. The Laplacian is of interest when studying signals defined on a graph. Consider a sequence of signals $\{\boldsymbol{x}_t\}$ indexed by $t \geq 0$. Given an initial condition $\boldsymbol{x}_0$, the solution to the heat equation on G is the sequence for continuous time $t > 0$ that satisfies

$$\frac{\partial}{\partial t}\boldsymbol{x}_t = -\alpha \boldsymbol{L}\boldsymbol{x}_t , \tag{1}$$

where the positive constant α represents the thermal diffusivity. Equivalently, in discrete time, we get

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \alpha \boldsymbol{L}\boldsymbol{x}_t . \tag{2}$$

Since all eigenvalues of $\boldsymbol{L}$ are non-negative, it follows that the matrix $\boldsymbol{T} = I - \alpha\boldsymbol{L}$ has one eigenvalue equal to 1, with corresponding eigenvector $\boldsymbol{f}_0$, and all other eigenvalues of $\boldsymbol{T}$ are less than 1 if $\alpha \leq 1$. Moreover, if G is not bipartite, then all other eigenvalues have magnitude strictly less than 1, and so it follows that $\boldsymbol{x}_t$ converges to $\|\boldsymbol{x}_0\|\, \boldsymbol{f}_0$ as $t \to \infty$, where $\|\boldsymbol{x}_0\|$ denotes the Euclidean norm of $\boldsymbol{x}_0$. From this perspective, $\boldsymbol{T}$ can be viewed as a smoothing or low-pass operator, and subsequently, $\boldsymbol{L}$ can be viewed as a high-pass operator.

Given a signal $\boldsymbol{x}$ on G, the quadratic form $\boldsymbol{x}^T\boldsymbol{L}\boldsymbol{x}$ also has an interesting interpretation. From the definition of $\boldsymbol{L}$, it directly follows that

$$\boldsymbol{x}^T\boldsymbol{L}\boldsymbol{x} = \sum_{(u,v) \in E} \left(\frac{x(u)}{\sqrt{\deg(u)}} - \frac{x(v)}{\sqrt{\deg(v)}} \right)^2 . \tag{3}$$

The suggests that $\boldsymbol{x}^T\boldsymbol{L}\boldsymbol{x}$ can be interpreted as the (degree-normalized) disagreement between the values at neighbouring nodes. Based on this observation, the dynamics (1) and (2) have been used in recent years to develop efficient algorithms for consensus-based in-network distributed processing [9, 10].

[2] A simple graph has no self-loops (i.e., all edges are between two distinct vertices) and no multiple edges (i.e., there is at most one edge between any pair of vertices).

[3] If G is not connected then the multiplicity of the eigenvalue 0 is equal to the number of connected components of G.

3. UNCERTAINTY PRINCIPLES FOR SIGNALS ON GRAPHS

Agaskar and Lu [3, 4] initiate the study of uncertainty principles for signals supported on graphs. The classical uncertainty principle states that a continuous-time signal defined on the unit interval cannot be simultaneously well-localized in both time and frequency. In signal processing one often wishes to obtain a parsimonious representation of a signal (e.g., through a linear transformation). The uncertainty principle has had profound implications for the design of bases which are well-localized in both time and frequency.

We follow the definitions of graph and spectral spread introduced in [3, 4]. The graph spread of a signal $\boldsymbol{x}$ is defined as

$$\Delta_g^2(\boldsymbol{x}) = \min_{v \in V} \frac{1}{\|\boldsymbol{x}\|^2} \sum_{u \in V} d(u,v)^2 x(u)^2 \ , \tag{4}$$

where $d\colon V \times V \to \mathbb{R}$ is any distance defined over the graph satisfying $d(u,v) \geq 0$, $d(u,u) = 0$, and $d(u,v) \geq 1$ if $u \neq v$. The spectral spread of a signal $\boldsymbol{x}$ is defined as

$$\Delta_s^2(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \ . \tag{5}$$

The uncertainty principle states that both graph and spectral spreads cannot be both made arbitrarily small for the same signal $\boldsymbol{x}$. For example, suppose $\boldsymbol{x}$ is such that $\Delta_g^2(\boldsymbol{x}) = 0$. This is only possible if there is a vertex v with $x(v) = 1$, and $x(u) = 0$ for all $u \neq v$, since $d(u,v) > 0$ for $u \neq v$. For such a signal [4], the spectral spread is $\Delta_s^2(\boldsymbol{x}) = 1$, and this can be seen directly from (3).

Now, let us also consider the other extreme, where the spectral spread is zero. In this case we can show that:

Proposition 1. *Let G be connected and let $\boldsymbol{x}$ be a signal on G such that $\Delta_s^2(\boldsymbol{x}) = 0$. Then $\Delta_g^2(\boldsymbol{x}) \geq 1/2$.*

Proof. Since we assume the graph is connected, the Laplacian L has an eigenvalue $\lambda_0 = 0$ with multiplicity one, and thus spectral spread is zero when only the first coordinate of $\boldsymbol{F}^T \boldsymbol{x}$ is nonzero. As a consequence, if $\Delta_s^2(\boldsymbol{x}) = 0$ then $\boldsymbol{x} \propto \boldsymbol{f}_0$ and the graph spread is

$$\Delta_g^2(\boldsymbol{x}) = \min_{v \in V} \frac{\sum_{u \in V} d(u,v)^2 \deg(u)}{\sum_{v' \in V} \deg(v')} \ .$$

Since $d(u,v) \geq 1$ for $u \neq v$,

$$\Delta_g^2(\boldsymbol{x}) \geq \min_{v \in V} \frac{\sum_{u \neq v} \deg(u)}{\sum_{v' \in V} \deg(v')} \ ,$$

and taking $v_{\max} \in \operatorname{argmax}_u \deg(u)$, we obtain

$$\Delta_g^2(\boldsymbol{x}) \geq \frac{\sum_{u \neq v_{\max}} \deg(u)}{\sum_{v' \in V} \deg(v')} \ .$$

Note that $\deg(v_{\max}) \leq n-1$ and

$$\sum_{v' \in V} \deg(v') \leq (n-1) + \sum_{u \neq v_{\max}} \deg(u) \ ,$$

and so

$$\begin{aligned} \Delta_g^2(\boldsymbol{x}) &\geq 1 - \frac{n-1}{n-1+\sum_{u \neq v_{\max}} \deg(u)} \\ &\geq \frac{1}{2}, \end{aligned}$$

where the last inequality follows from noting that $\deg(u) \geq 1$ because G is connected. □

We note that this lower bound is achieved for the star graph [4].

Combining our Proposition 1 with the observation in [4] that $\Delta_g^2(\boldsymbol{x}) = 0$ implies $\Delta_s^2(\boldsymbol{x}) = 1$ illustrates that some form of an uncertainty principle exists for signals on graphs, in the sense that a signal with zero graph spread must have non-zero spectral spread, and vice versa. In general, it is of interest to quantify the extent to which a signal can be localized in the graph domain while having a given spectral spread. Specifically, we would like to characterize the lower uncertainty curve $\gamma(s)$ defined as [4]

$$\gamma(s) = \min_{\boldsymbol{x}\colon\, \Delta_s^2(\boldsymbol{x}) = s} \Delta_g^2(\boldsymbol{x}) \ .$$

The spectral spread (5) is directly expressed as a Rayleigh quotient in terms of the Laplacian. The graph spread (4) can also be expressed as a Rayleigh quotient [4] by defining $n \times n$ diagonal matrices $\boldsymbol{P}_v^2$ with $[\boldsymbol{P}_v^2]_{u,u} = d(u,v)^2$; then

$$\Delta_g^2(\boldsymbol{x}) = \min_{v \in V} \frac{\boldsymbol{x}^T \boldsymbol{P}_v^2 \boldsymbol{x}}{\|\boldsymbol{x}\|^2} \ .$$

There is a direct connection between Rayleigh quotients and the eigenvalues of a matrix via the Courant-Fischer theorem [11]. This motivates studying the Laplacian spectrum of small-world networks as a step towards characterizing the lower uncertainty curve of small-world networks.

4. LAPLACIAN SPECTRUM OF RING GRAPHS

Towards characterizing the Laplacian spectrum of small-world networks, in this section we discuss the Laplacian spectrum of K-connected rings. Recall that we index the vertices using the integers modulo n; i.e., $V = \{0, 1, \ldots, n-1\}$. For an integer $K \geq 1$, the K-connected ring on n vertices is a graph G which has an edge between vertices u and v if and only if

$$u - v \pmod n \leq K \ .$$

Taking $K = 1$ gives the standard ring topology where each vertex v has two neighbours, $v+1$ and $v-1$, with arithmetic taken modulo n.

Observe that the adjacency matrix $\boldsymbol{A}_K$ of the K-connected ring is circulant. Also note that every vertex has exactly $2K$ neighbours, so the degree matrix is $\boldsymbol{D}_K = (2K)\boldsymbol{I}$. Consequently, the matrix $\boldsymbol{T}_K = \boldsymbol{D}_K^{-1/2} \boldsymbol{A}_K \boldsymbol{D}_K^{-1/2}$ is circulant with entries

$$[\boldsymbol{T}_K]_{u,v} = \begin{cases} \frac{1}{2K} & \text{if } u - v \pmod n \leq K \ , \\ 0 & \text{otherwise.} \end{cases}$$

It follows from well-known properties of circulant matrices [12] that the eigenvalues of $\boldsymbol{T}_K$ are given (in no particular order) by

$$\lambda_j(\boldsymbol{T}_K) = \frac{1}{K} \sum_{k=1}^{K} \cos(2\pi jk/n) \ ,$$

for $j = 0, 1, \ldots, n-1$, with corresponding eigenvectors $\boldsymbol{f}_j$ having entries

$$f_j(v) = e^{2\pi i v j/n} \ , \tag{6}$$

where $i = \sqrt{-1}$ is the imaginary unit. Note that, for any K-connected ring, for $j \leq \lceil \frac{n}{2} \rceil$ the eigenvalues $\lambda_j(\boldsymbol{T}_K)$ and $\lambda_{n-j}(\boldsymbol{T}_K)$ are equal, since $\cos(2\pi c/n) = \cos(2\pi(n-c)/n)$ for $0 \leq c \leq n-1$.

Since $\boldsymbol{L}_K = \boldsymbol{I} - \boldsymbol{T}_K$, the eigenvalues of $\boldsymbol{L}_K$ are simply given by $1 - \lambda_j(T_K)$.

5. LAPLACIAN SPECTRUM OF SMALL-WORLD NETWORKS

Next we study the Laplacian spectrum of small-world networks. The small-world model, introduced by Watts and Strogatz [5], defines an ensemble of random graphs parameterized by an integer $K \geq 1$ and a rewiring probability $p \in [0, 1]$. A realization of a small-world random graph on n nodes with parameters K and p is obtained using the following procedure. Begin with a K-connected ring on n nodes. Then, for each edge $(u, v) \in E$, independently and with probability p, rewire the edge by re-connecting one of the vertices, say u, to a new vertex chosen uniformly from $V \setminus \{u\}$.

Watts and Strogatz [5] observed that many networks are both highly clustered and have short average path lengths. They are highly clustered in the sense that, on average over all nodes $v \in V$, if v_1 and v_2 are both neighbours of v, then v_1 and v_2 are also neighbours in a significant proportion of cases. Networks have short average path length if, taking the average over all pairs of vertices $(v_1, v_2) \in V \times V$, the length of the shortest path between v_1 and v_2 is significantly shorter than n; e.g., it may be of order $\mathcal{O}(\log n)$. Watts and Strogatz [5] proposed the model described above under the motivation of generating random graphs which exhibit these properties.

In the small-world model, when $p \approx 0$, then very few edges are re-wired and the graph is essentially still a K-connected ring. The K-connected ring is well-clustered, in the sense that many of each node's $2K$ neighbours will also be neighbours. However, the average path-length is quite high (i.e., of the order $\mathcal{O}(n/K)$). On the other hand, when $p \approx 1$, then all edges are rewired and the small-world model is similar to an Erdős-Rényi random graph [13]; that is, one where for each pair of vertices, the edge (u, v) is present in the graph independently and with probability $p' = \frac{2K}{n-1}$. Such graphs are known to have have short path length, but they are not highly clustered [14]. Watts and Strogatz [5] observed that, interestingly, for intermediate values of the rewiring probability p, the resulting random graphs exhibit both high clustering and short average path length.

In this section we study the spectrum of the expected Laplacian of a small-world graph. Through experiments we have observed that the spectrum of small-world graphs is tightly concentrated around its expectation. Let us fix the model parameters $K \geq 1$ and $p \in [0, 1]$. Let $\boldsymbol{A}_K$ denote the adjacency matrix of a K-connected ring, and let $\overline{\boldsymbol{A}}$ denote the expected adjacency matrix of a small-world graph. Since each edge is rewired with probability p, and the rewiring is done in an independent and uniform manner, it follows that the expected adjacency matrix is given by

$$\overline{\boldsymbol{A}} = \boldsymbol{A}_K - p\boldsymbol{A}_K + \frac{2pK}{n-1}(\mathbf{1}\mathbf{1}^T - \boldsymbol{I}) \ .$$

After rewiring, the expected degree of each node is still $2K$, and so the expected degree matrix is $\overline{\boldsymbol{D}} = (2K)\boldsymbol{I}$. As a surrogate for the expected diffusion matrix, we take $\overline{\boldsymbol{T}} = \overline{\boldsymbol{D}}^{-1/2}\overline{\boldsymbol{A}\boldsymbol{D}}^{-1/2}$ which is equal to

$$\overline{\boldsymbol{T}} = \boldsymbol{T}_K + p\left(\frac{1}{n-1}(\mathbf{1}\mathbf{1}^T - \boldsymbol{I}) - \boldsymbol{T}_K\right) \ , \tag{7}$$

where $\boldsymbol{T}_K$ is the diffusion matrix of the K-connected ring.

We will obtain approximate expressions for the eigenvalues of $\overline{\boldsymbol{T}}$ by using concepts from matrix perturbation theory. To this end, we recall some definitions and basic facts from [15]. Matrix perturbation theory involves the study of eigenvalues of a matrix $\boldsymbol{T}(\epsilon)$ which has been perturbed from a baseline matrix $\boldsymbol{T}_0$ by a small amount $\epsilon > 0$. If $\boldsymbol{T}(\epsilon)$ is a continuous function of ϵ, then the eigenvalues of $\boldsymbol{T}(\epsilon)$ are continuous functions of ϵ too [15]. An eigenvalue of $\boldsymbol{T}(\epsilon)$ is called *stable* if it does not depend on ϵ. An eigenvalue of $\boldsymbol{T}(\epsilon)$ is called *semi-simple* if its algebraic multiplicity is equal to its geometric multiplicity.

Now, consider the special case where $\boldsymbol{T}(\epsilon)$ is a linear perturbation of $\boldsymbol{T}_0$; i.e., $\boldsymbol{T}(\epsilon) = \boldsymbol{T}_0 + \epsilon\boldsymbol{E}$ for some perturbation matrix $\boldsymbol{E}$. Suppose that λ_0 is a semi-simple double eigenvalue of $\boldsymbol{T}_0$ with corresponding eigenvectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. Then for $\epsilon > 0$, $\boldsymbol{T}(\epsilon)$ has two eigenvalues, $\lambda_{0,1}$ and $\lambda_{0,2}$, related to λ_0, which can be given in the form of a power series,

$$\lambda_{0,1} = \lambda_0 + \epsilon\lambda' + o(\epsilon)$$
$$\lambda_{0,2} = \lambda_0 + \epsilon\lambda'' + o(\epsilon),$$

where λ' and λ'' are the eigenvalues of the 2×2 matrix

$$\begin{bmatrix} \boldsymbol{f}_1^T\boldsymbol{E}\boldsymbol{f}_1 & \boldsymbol{f}_1^T\boldsymbol{E}\boldsymbol{f}_2 \\ \boldsymbol{f}_2^T\boldsymbol{E}\boldsymbol{f}_1 & \boldsymbol{f}_2^T\boldsymbol{E}\boldsymbol{f}_2 \end{bmatrix} .$$

To simplify the discussion for this short paper, we focus on the case where n is odd. Then the diffusion matrix $\boldsymbol{T}_K$ of the K-connected ring on n nodes has exactly one eigenvalue $\lambda_0(\boldsymbol{T}_K) = 1$, and all other eigenvalues come in pairs

$$\lambda_j(\boldsymbol{T}_K) = \lambda_{n-j}(\boldsymbol{T}_K) = \frac{1}{K}\sum_{k=1}^{K}\cos(2\pi jk/n) \ ,$$

for $j = 1, \ldots, \lceil \frac{n}{2} \rceil$, and the eigenvectors are exactly those described by (6). We will use the results from matrix perturbation theory mentioned above to obtain expressions for the eigenvalues of $\overline{\boldsymbol{T}}$ by taking $\boldsymbol{T}_0 = \boldsymbol{T}_K$, $\epsilon = p$, and

$$\boldsymbol{E} = \frac{1}{n-1}(\mathbf{1}\mathbf{1}^T - \boldsymbol{I}) - \boldsymbol{T}_K \ .$$

It is easy to verify that $\lambda_0(\overline{\boldsymbol{T}}) = 1$ is a stable eigenvalue of $\overline{\boldsymbol{T}}$ since $\boldsymbol{f}_0 = (1/\sqrt{n})\mathbf{1}$ is a constant vector. All remaining eigenvalues of $\boldsymbol{T}_K$ have multiplicity two. For $j = 1, \ldots, \lceil \frac{n}{2} \rceil$, the eigenvalue $\lambda_j(\boldsymbol{T}_K)$ has eigenvectors $\boldsymbol{f}_j$ and $\boldsymbol{f}_{n-j}$. Note, also, that $\boldsymbol{f}_j^T\mathbf{1} = \boldsymbol{f}_{n-j}^T\mathbf{1} = 0$ since the eigenvectors are orthogonal and $\boldsymbol{f}_0$ is proportional to $\mathbf{1}$. Using these facts one can verify that

$$\boldsymbol{f}_j^T\boldsymbol{E}\boldsymbol{f}_j = \boldsymbol{f}_{n-j}^T\boldsymbol{E}\boldsymbol{f}_{n-j} = \frac{1}{n-1} - \lambda_j(\boldsymbol{T}_K)$$

and

$$\boldsymbol{f}_j^T\boldsymbol{E}\boldsymbol{f}_{n-j} = \boldsymbol{f}_{n-j}^T\boldsymbol{E}\boldsymbol{f}_j = 0 \ .$$

Thus, for the small-world model, the eigenvalues of $\overline{\boldsymbol{T}}$ are $\lambda_0(\overline{\boldsymbol{T}}) = 1$ and, for $j = 1, \ldots, \lceil \frac{n}{2} \rceil$,

$$\lambda_j(\overline{\boldsymbol{T}}) = (1-p)\lambda_j(\boldsymbol{T}_K) + \frac{p}{n-1} + o(p),$$

with multiplicity two, and with the same eigenvectors $\boldsymbol{f}_j$ and $\boldsymbol{f}_{n-j}$.

6. CONCLUSION AND FUTURE WORK

We prove a lower bound on the graph spread of signals with zero spectral spread, and we provide an approximate characterization of the Laplacian spectrum of small-world networks. Our future work involves more precisely characterizing the shape of the lower uncertainty curve for ring graphs, small-world networks, and more general graphs using the results presented here.

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